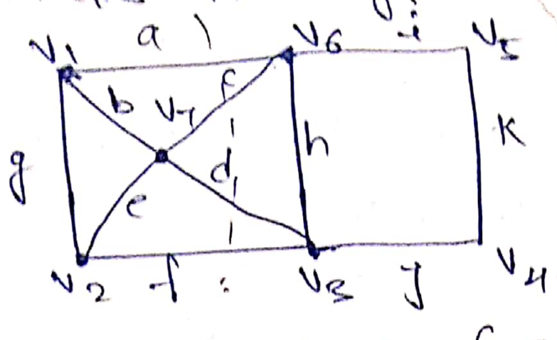
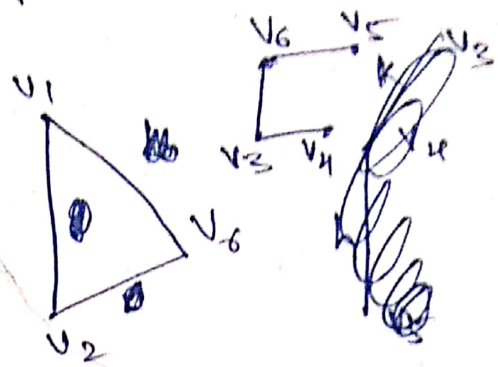
Cut-Sets

A cut-set S of a connected graph G is a minimal set of edges of G whose removal from G disconnects G, i.e. removal of no proper subset of S disconnects G. e.g.:



G_1



G_2

Cut set $S = \{a, c, d, f\}$

Note: A cut-set always cuts a graph into two.

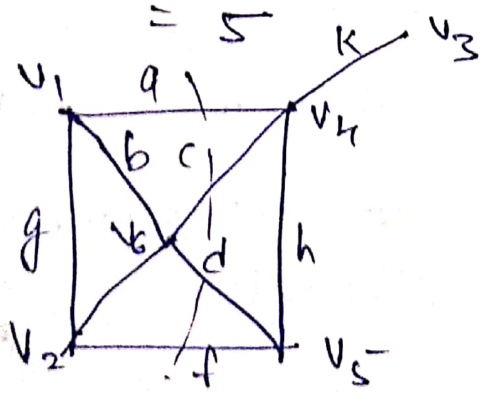
$\therefore$ a cut-set can also be defined as a minimal set of edges in a connected graph whose removal reduces the rank of the graph by one.

$r(G_1) = 7-1 = 6$, $\because n = 7, k = 1$

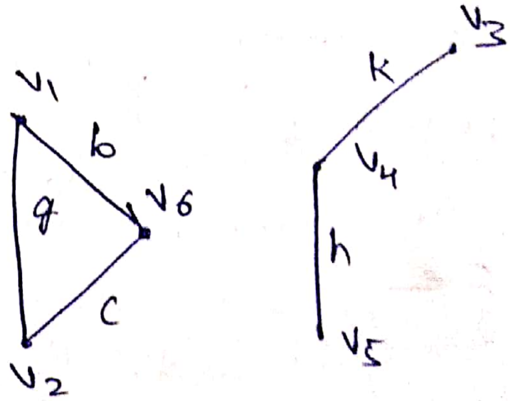
$r(G_2) = 3+4-2$, $\because n = 3+4 = 7, k = 2$

$= 5$

Ex →



G_1



G_2

$r(G_1) = 6-1 = 5$

$r(G_2) = 3+3-2 = 4$

Another Def^n → ~~A cut-set is a minimal no. of edges~~ (1a)

~~Almost~~. all the vertices of

If we partition G into two mutually exclusive subsets, a cut-set is a minimal no. of edges whose removal from G destroys all paths b/w these two set of vertices. e.g. in 2nd example above cut-set {a, c, d, f} connects vertex set {v_1, v_2, v_6} with {v_3, v_4, v_5}

Note → In a tree, removal of any edge e breaks the tree into two parts, every edge of a tree is a cut-set.

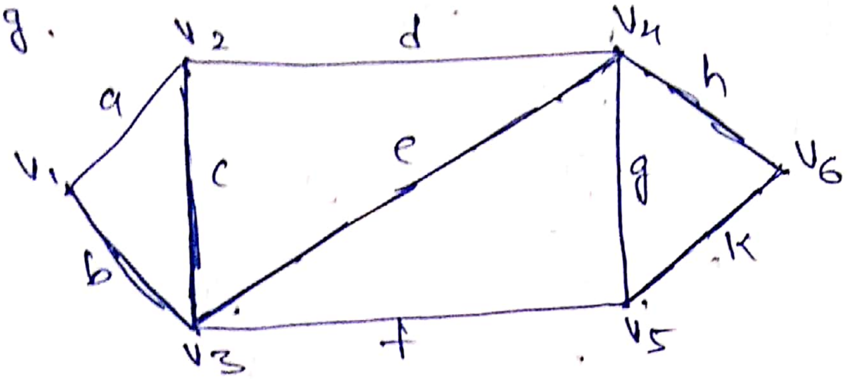
Note → Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

Note → In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set

Note → Every circuit has an even number of edges in common with any cut-set.

Fundamental Cut-Set → Consider a S.T. 'T' of a connected graph G. Take any branch 'b' in T. ∵ {b} is a cut-set in T, {b} partitions all vertices of T into two disjoint sets — one at each end of b. Consider the same partition of vertices in G and the cut-set S in G that corresponds to this partition. Cut-set S will contain only one branch 'b' of T and rest of the edges in S are chords w.r.t. 'T'. Such a cut-set S containing exactly one branch of a tree 'T' is called a fundamental cut-set w.r.t. 'T'. Also called a basic cut-set.

e.g.



S.T. → edges = b c e h K

Cut Sets = {a, c}; {a, b}, {d, e, f}, {h, g, f}, {k, f}

→ all are fundamental cut-sets w.r.t. 'T'

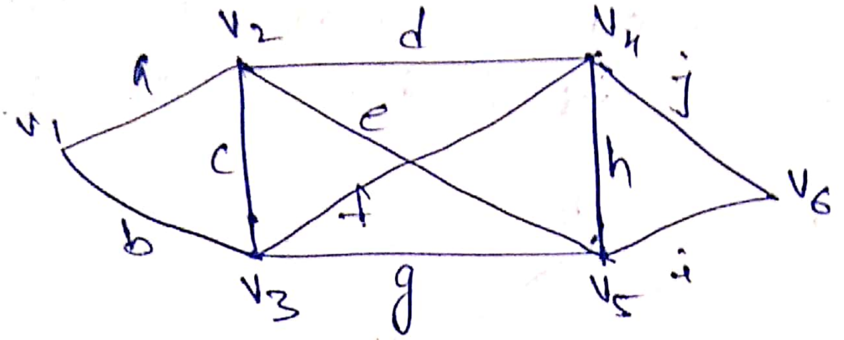
Just as every chord of a S.T. defines a fundamental circuit, every branch of a S.T. defines a unique fundamental cut-set.

<u>Note</u> - The fundamental cut-set has meaning only w.r.t. a given S.T.

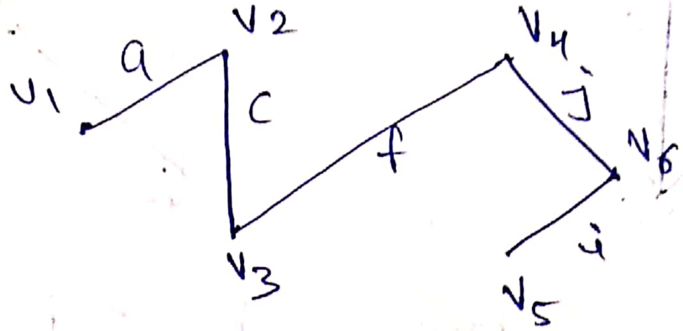
<u>Note</u> - w.r.t. a given S.T., a chord c_i, that determines a fundamental circuit 'C' occurs in every fundamental cut-set associated with the branches in C and in no other.

Note→ w.r.t. a given S.T. 'T', a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S and in no others.

Ex→ Consider—



S.T. →



branches = a, c, f, j, i

chords = b, g, h, d, e

Note→ An Euler graph G can not have a cut-set of odd no. of edges.

Connectivity and Separability

Connectivity

Edge Connectivity

- Each cut-set of a connected graph G consists of certain no. of edges. The no. of edges in the smallest cut-set is defined as the edge connectivity of G. i.e. the min. no. of edges whose removal reduces the rank of graph by one.
- edge conn. of a tree is one

Vertex Connectivity

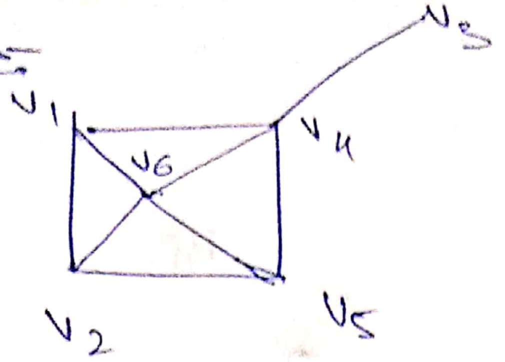
- min. no. of vertices whose removal from G leaves the graph disconnected
- vertex conn. of a tree is one.

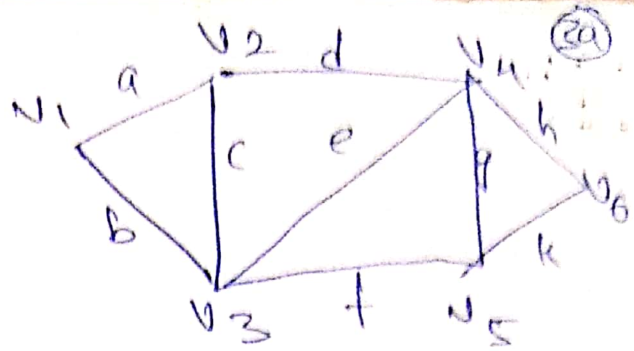
Separable Graph

A conn. graph is said to be separable if its vertex conn. is one. All other connected graphs are called nonseparable

- A conn. gph. G is said to be separable if ∃ a subgph g in G s.t. $\bar{g}$ and g have only one vertex in common.
- The vertex whose removal disconnects the graph is called cut-vertex, cut-node or an articulation point
- In a tree, every vertex with degree greater than one is a cut-vertex.

Examples-





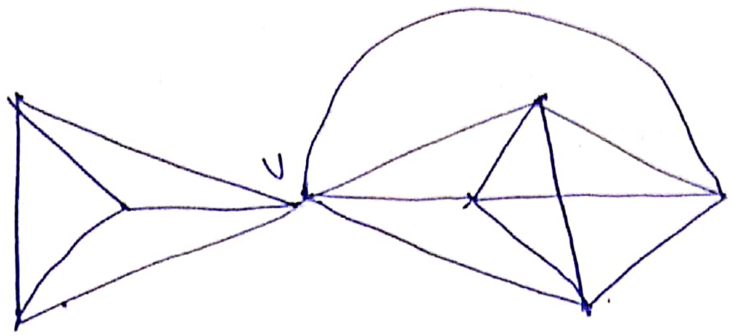
edge connectivity = 1

V.C. = 1 → separable gph.

e.c. = 2

V.C. = 2

— nonseparable



e.c. = 3 , V.C. = 1 , v → cut-vertex

separable gph.

Note- A vertex v in a conn. gph G is a cut-vertex iff ∃ two vertices x and y in G s.t. every path b/w x & y passes thru v.

Note- The edge c. of a gph G can not exceed the degree of the vertex with the smallest degree in G.

Note- The vertex c. of any gph. G can never exceed the e.c. of G.

Pf- Let α denote the e.c. of G. ∴ ∃ a cut-set S in G with α edges. Let S partition the vertices of G into subsets V_1 and V_2. By removing at most α vertices from V_1 (or V_2) on which the edges in S are incident (together with all the edges incident on these vertices), we can effect the removal of S from G. Hence the thm.

Note – Every cut-set in a nonseparable gph. with more than two vertices contains at least two edges.

Note – The maximum v.c. one can achieve with a graph G of n vertices and e edges ($e \geq n-1$) is the integral part of the number $\frac{2e}{n}$ i.e. $\left[\frac{2e}{n}\right]$

Results →

$$v.c. \leq e.c \leq \frac{2e}{n}$$

and

$$\text{max. v.c. possible} = \left[\frac{2e}{n}\right]$$

e.g. consider a $G(V,E)$, $n(V) = 8$, $n(E) = 16$

one can achieve a v.c. as high as

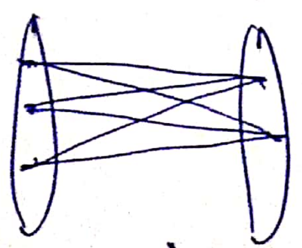
$$\left[\frac{2\times 16}{8}\right] = [4] = 4$$

Defn – A gph G is said to be k-connected if the v.c. of G is K. ∴ a 1-connected gph is the same as sep. gph.

Ex – Consider $K_{m,n}$ → complete bipartite gph.

~~first~~ e.c. = $\min\{m,n\}$

e.g. $K_{3,2}$, e.c. = $\min\{3,2\} = 2$



Ex → E.c. of a complete gph with n vertices is $(n-1)$